# Irreducible complexity of iterated symmetric bimodal maps

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Abstract. We introduce a tree structure for the iterates of symmetric bimodal maps and identify a subset which we prove to be isomorphic to the family of unimodal maps. This subset is used as a second factor for a $*$-product that we define in the space of bimodal kneading sequences. Finally, we give some properties for this product and study the $*$-product induced on the associated Markov shifts.

## 1. Introduction and preliminary definitions

The concept of irreducible complexity of a biological system was introduced by Behe, [**1**], in 1996. His point of view is that an organim consisting of a finite, possibly very large, number of independent components, coupled together in some way, exhibits irreducible complexity if, by removing any of its component, the reduced system no longer functions meaningfully. Using the language of non-linear dynamics and chaos theory, Boyarsky and Góra, [**2**], reinterpreted Behe's definition from a Markov transition matrix perspective by saying that a system is irreducibly complex if the associated transition matrix is primitive but no principal submatrix is primitive.

It is our conviction that the concept of reducible complexity of a dynamical system can also be interpreted in terms of a factorization: within Milnor and Thurston's kneading theory framework, and the topological classification obtained from it, Derrida, Gervois, and Pomeau, [**4**], introduced a $\star$-product between unimodal kneading sequences for which it was possible to prove that the topological entropy, a measure of complexity, of a factorizable system is equal to the topological entropy of one of the factors. Despite of a larger number of its components, the complexity of the system remains the same whenever its irreducible component, a factor of the product, does not change.

Some years latter, Lampreia, Rica da Silva, and Sousa Ramos, [**6**], introduced a Markov transition matrix formalism associated with the kneading theory and a product between unimodal matrices corresponding to the Derrida, Gervois, and Pomeau $\star$-product. Then, they proved that irreducible unimodal kneading sequences corresponds to primitive Markov transition matrices.

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With this work we would like to introduce the generalization, for bimodal symmetric maps of the interval, of the $\star$-product and the corresponding product between transition matrices.

Consider a two-parameter family $f_{a,b}$ of maps, from the closed interval $I = [c_0, c_3]$ into itself, with two critical points, usually called a bimodal family of maps of the interval, see [**3**], [**11**], [**7**]. Once fixed the parameters $(a, b)$, the map $f_{a,b}$ is piecewise monotone and hence $I$ can be subdivided in the following three subintervals: $L = [c_0, c_1]$, $M = [c_1, c_2]$ and $R = [c_2, c_3]$, where $c_i$ are the critical points or the extremal points, in such a way that the restriction of $f$ to each interval is strictly monotone. We will choose the family of maps such that the restrictions $f_{a,b|L}$ and $f_{a,b|R}$ are increasing and the restriction $f_{a,b|M}$ is decreasing.

For each value $(a, b)$ we define the orbits of the critical points by:

$$O(c_i) = \{x_j : x_j = f^j(c_i),\ j \in \mathbb{N}\}$$

with $i = 1, 2$.

With the aim of studying the topological properties of these orbits we associate to each orbit $O(c_i)$ a sequence of symbols $S = S_1S_2 \ldots S_j \ldots$ where $S_j = L$ if $f^j_{a,b}(c_i) < c_1$, $S_j = A$ if $f^j_{a,b}(c_i) = c_1$ , $S_j = M$ if $c_1 < f^j_{a,b}(c_i) < c_2$, $S_j = B$ if $f^j_{a,b}(c_i) = c_2$ and $S_j = R$ if $f^j_{a,b}(c_i) > c_2$. If we denote by $n_M$ the frequency of the symbol $M$ in a finite subsequence of $S$ we can define the $M$-parity of this subsequence according to whether $n_M$ is even or odd. In what follows (see [**11**]) we define an order relation in $\Sigma_5 = \{L, A, M, B, R\}^{\mathbb{N}}$ that depends on the $M$-parity.

Let $V$ be a vector space of three dimension defined over the rationals having as a basis the formal symbols $\{L, M, R\}$, then to each sequence of symbols $S = S_1S_2 \ldots S_j \ldots$ we can associate a sequence $\theta = \theta_0 \ldots \theta_j \ldots$ of vectors from $V$, setting $\theta_j = \prod_{i=0}^{j-1} \epsilon(S_i)S_j$ with $j > 0, \theta_0 = S_0$ when $i = 0$ and $\epsilon(L) = -\epsilon(M) = \epsilon(R) = 1$, where to the symbols corresponding to the critical points $c_1$ and $c_2$ we associate the vector $\frac{L+M}{2}$ and $\frac{M+R}{2}$. Thus $\epsilon(A) = \epsilon(B) = 0$. Choosing then a linear order in the vector space $V$ in such a way that the base vectors satisfy $L < M < R$ we are able to order the sequence $\theta$ lexicographically, that is, $\theta < \bar{\theta}$ iff $\theta_0 = \bar{\theta}_0, \ldots, \theta_{j-1} = \bar{\theta}_{j-1}$ and $\theta_j < \bar{\theta}_j$ for some integer $i \geq 0$. Finally, introducing $t$ as an undetermined variable and taking $\theta_j$ as the coefficients of a formal power series $\theta$ (invariant coordinate) we obtain $\theta = \theta_0 + \theta_1 t + \ldots = \sum_{j=0}^{\infty} \theta_j t^j$.

The sequences of symbols corresponding to periodic orbits of the critical points $c_1$ and $c_2$ are $P = AP_1P_2 \ldots P_{p-1}A \ldots$ and $Q = BQ_1Q_2 \ldots Q_{q-1}B \ldots$. In what follows we denote by $P^{(p)} = P_1P_2 \ldots P_{p-1}A$ and $Q^{(q)} = Q_1Q_2 \ldots Q_{q-1}B$ the periodic blocks associated to $P$ and $Q$. The realizable itineraries of the critical points $c_1$ and $c_2$ for the maps previously defined are called by kneading sequences [**11**].

## 2. Symbolic dynamics for symmetric bimodal maps

Denote by $\mathcal{F}_{KS}$ the set of pairs of kneading sequences $(P, Q)$, with $(P, Q)$ either a pair of stable orbits, or a doubly stable orbit. In Table 1, we give the subset of kneading sequences, with length $p, q < 5$.

Table 1. Kneading data for bimodal maps (detail)

| | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 1 | | | | | | | | | | | | | | | | | | | | * | |
| 2 | | | | | | | | | | | | | | | | * | | * | | * | |
| 3 | | | | | | | | * | | * | | | | | | * | | * | | * | |
| 4 | | | | | | | * | | * | | | | * | | | | * | | * | | * |
| 5 | | | | | $\circledast$ | * | | * | | * | | | | | | * | | * | | * | |
| 6 | | | | | * | $\circledast$ | | * | | * | | | | | | * | | * | | * | |
| 7 | | | | * | | | $\circledast$ | | * | | * | | * | | * | | * | | * | | * |
| 8 | | | * | | * | * | | $\circledast$ | | * | | | | | | * | | * | | * | |
| 9 | | | | * | | | * | | $\circledast$ | | * | | * | | * | | * | | * | | * |
| 10 | | | * | | * | * | | * | | $\circledast$ | | | | | | * | | * | | * | |
| 11 | | | | | | | * | | * | | $\circledast$ | | * | | * | | * | | * | | * |
| 12 | | | | | | | | | | | | | | | | * | | * | | * | |
| 13 | | | | * | | | * | | * | | * | | $\circledast$ | | * | | * | | * | | * |
| 14 | | | | | | | | | | | | | | $\circledast$ | | * | | * | | * | |
| 15 | | | | | | | * | | * | | * | | * | | $\circledast$ | | * | | * | | * |
| 16 | | * | * | | * | * | | * | | * | | * | | * | | $\circledast$ | | * | | * | |
| 17 | | | | * | | | * | | * | | * | | * | | * | | $\circledast$ | | * | | * |
| 18 | | * | * | | * | * | | * | | * | | * | | * | | * | | $\circledast$ | | * | |
| 19 | | | | * | | | * | | * | | * | | * | | * | | * | | $\circledast$ | | * |
| 20 | * | * | * | | * | * | | * | | * | | * | | * | | * | | * | | $\circledast$ | |
| 21 | | | | * | | | * | | * | | * | | * | | * | | * | | * | | $\circledast$ |

Legend: For the lines of the table, we have:

| |
|---|
| 1-RLLA, 2-RLA, 3-RLMA, 4-RLB, 5-RA, 6-RMRA, 7-RMB, 8-RMMA, 9-RMMB,10-RMA,11-RMLB, |
| 12-RMLA, 13-RB, 14-RRLA, 15-RRLB, 16-RRA, 17-RRMB, 18-RRMA, 19-RRB, 20-RRRA, 21-RRRB |

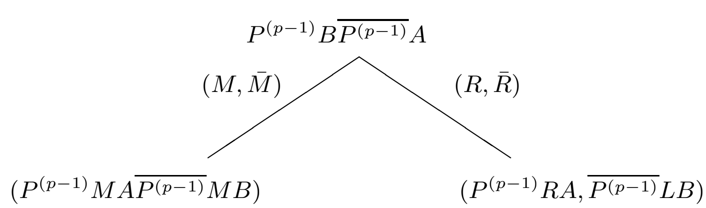
The corresponding columns are given by the conjugate of the previous sequences.

We define a tree $\mathcal{D}$ that corresponds to the diagonal in $\mathcal{F}_{KS}$ and codify the symmetric bimodal maps. Each element $S \in \mathcal{D}$ is from one of the following types: $S$ is a pair of stable orbits, i.e., $S = (P, \overline{P}) = (P^{(p-1)}A, \overline{P^{(p-1)}}B)$; otherwise, $S$ is a doubly stable orbit, i.e., $S = P^{(p-1)}B\overline{P^{(p-1)}}A$, where $\overline{P^{(p-1)}} = \overline{P_1}\,\overline{P_2}...\overline{P_{p-1}}$ with $\overline{P_i} = R$ if $P_i = L$, $\overline{P_i} = M$ if $P_i = M$ and $\overline{P_i} = L$ if $P_i = R$, and $1 \le i \le p-1$.

Note that the set $\mathcal{D}$ is ordered considering the order of the sequences $P$ (or the inverse order in $\bar{P}$) induced by the order on the symbols $-R < -B < -M < -A < -L < L < A < M < B < R$.

Let $\mathcal{D}_1$ a subset of $\mathcal{D}$ with elements between $(M^\infty, M^\infty)$ and $(RM^\infty, LM^\infty)$, see Figure 1. Let $S^{(2p)} = (P^{(p-1)}A, \overline{P^{(p-1)}}B)$ or $S^{(2p)} = P^{(p-1)}B\overline{P^{(p-1)}}A$ and consider a full tree $\mathcal{T}$ which its elements are also between $(M^\infty, M^\infty)$ and $(RM^\infty, LM^\infty)$ and characterized by each vertex branch in two edges following the next rule:

Alternatively the vertices in each level of the tree are doubly stable $P^{(p-1)}B\overline{P^{(p-1)}}A$ or pairs of stable orbits $(P^{(p-1)}A, \overline{P^{(p-1)}}B)$. The doubly stable orbits occur in odd levels and the pairs of stable orbits in even levels. For the doubly stable orbit $P^{(p-1)}B\overline{P^{(p-1)}}A$ and according to the $M-$parity of $P^{(p-1)}$ is even or odd than the branching order can be described respectively by:



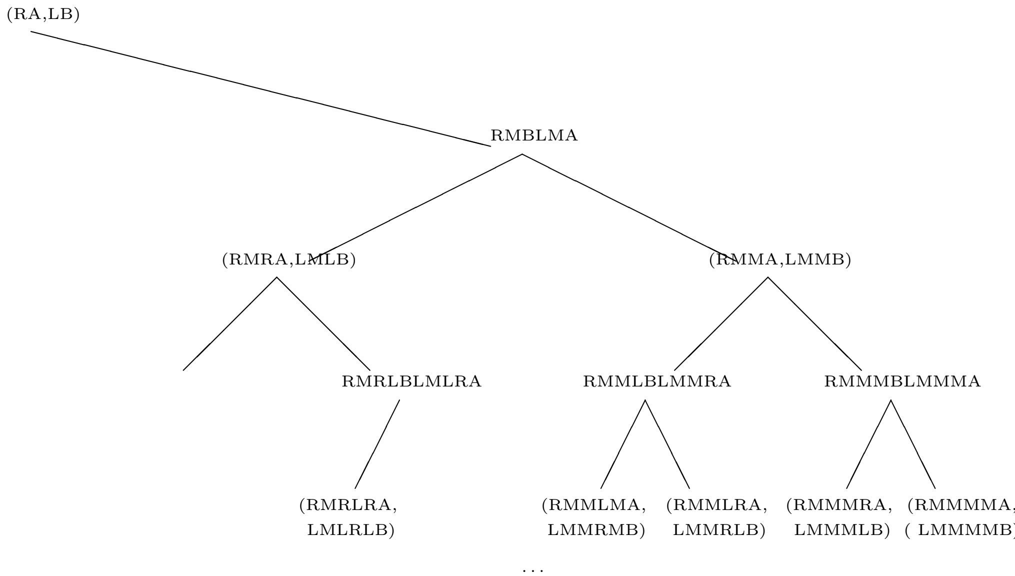
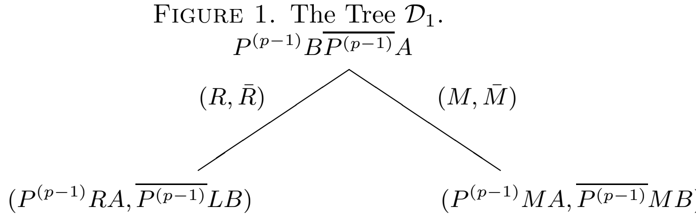
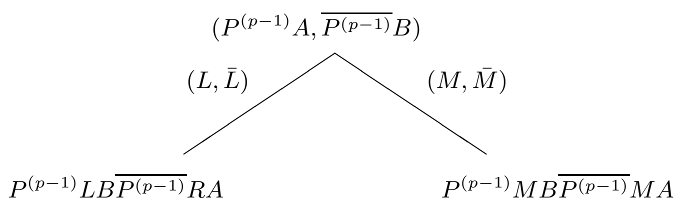
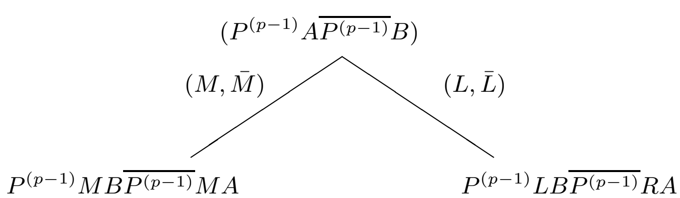



FIGURE 1. The Tree $\mathcal{D}_1$.




For the pairs of stable orbits the branching order can be described by:




according to the $M-$parity of $P^{(p-1)}A$ is respectively even or odd.

Using these rules we get, as mentioned before, the full tree $\mathcal{T}$, see Figure 2. The next result establish that to each $S=(P^{(p-1)}A,\overline{P^{(p-1)}}B)$, or

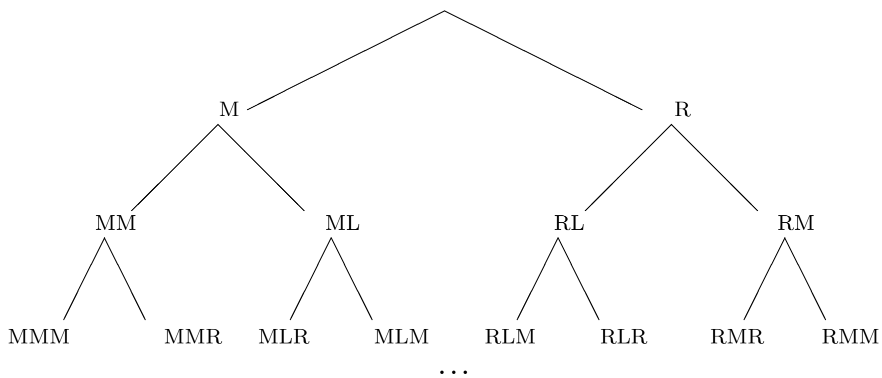
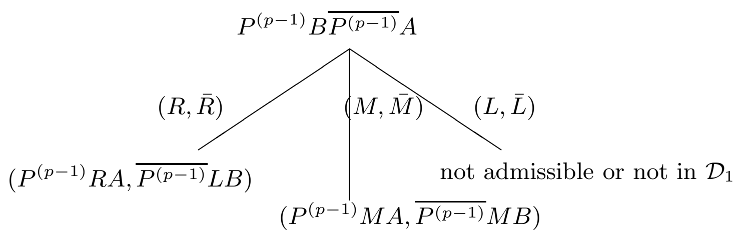



FIGURE 2. The Tree $\mathcal{T}$

$P^{(p-1)}B\overline{P^{(p-1)}}A$ in $\mathcal{D}_1$, corresponds a sequence $P^{(p-1)}$ in $\mathcal{T}$.

LEMMA 1. *If $S \in \mathcal{D}_1$, then $P^{(p-1)} \in \mathcal{T}$.*

PROOF. Let $S = P^{(p-1)}B\overline{P^{(p-1)}}A \in \mathcal{D}_1$ be a doubly stable orbit (odd level), with odd $M$-parity. Then, we have:




The doubly stable orbit $P^{(p-1)}B\overline{P^{(p-1)}}A$ leads, on the next level, to the pairs of stable orbits given by $(P^{(p-1)}XA, \overline{P^{(p-1)}}\bar{X}B) = (RM\ldots XA, LM\ldots \bar{X}B)$. Note that when $(X, \bar{X}) = (L, \bar{L})$ then

$$\sigma^{(p-1)}(LM\ldots \bar{X}B) = \sigma^{(p-1)}(LM\ldots RB) = RB\ldots > RM\ldots$$

which is not admissible or is not in $\mathcal{D}_1$. In the same way, the doubly stable ones obtained from pairs of stable orbits follows the rule in $\mathcal{T}$ because now the branch associated to $(R, \bar{R})$ is not admissible. The proof is analogous for the case when the $M$-parity of $P^{(p-1)}A$ is even. ∎

In what follows we denote by $\mathcal{T}_{KS}$ the set of kneading sequences associated to unimodal maps. Then, we have:

THEOREM 1. *The tree $\mathcal{D}_1$ is isomorphic to $\mathcal{T}_{KS}$.*

PROOF. Let $\mathcal{E}$ be a complete tree with two symbols $\{L, R\}$ where we consider the $R$-parity. There exist an isomorphism between $\mathcal{T}$ and $\mathcal{E}$, where each symbol $L$ in $\mathcal{E}$ corresponds to a symbol $M$ in $\mathcal{T}$ and each symbol $R$ in $\mathcal{E}$ corresponds to a symbol $L$ or $R$ in $\mathcal{T}$ according to the $(k-1)$- level is even or odd, respectively. Thus the $R$-parity in $\mathcal{E}$ corresponds to the $(R+L)$-parity in $\mathcal{T}$ and so, we have:

$$\begin{tikzcd}[column sep=small] & P_1P_2\dots P_{k-1} \text{ with } \#(R+L) \text{ even} \arrow[dl, dash, "\mathrm{M}"'] \arrow[dr, dash, "\mathrm{R}"] & \\ P_1P_2\dots P_{k-1}M & & P_1P_2\dots P_{k-1}R \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & P_1P_2\dots P_{k-1} \text{ with } \#(R+L) \text{ odd} \arrow[dl, dash, "\mathrm{R}"'] \arrow[dr, dash, "\mathrm{M}"] & \\ P_1P_2\dots P_{k-1}R & & P_1P_2\dots P_{k-1}M \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small] & P_1P_2\dots P_{k-1} \text{ with } \#(R+L) \text{ even} \arrow[dl, dash, "\mathrm{M}"'] \arrow[dr, dash, "\mathrm{L}"] & \\ P_1P_2\dots P_{k-1}M & & P_1P_2\dots P_{k-1}L \end{tikzcd}$$

if $(k-1)$-level is even and

$$\begin{tikzcd}[column sep=small] & P_1P_2\dots P_{k-1} \text{ with } \#(R+L) \text{ odd} \arrow[dl, dash, "\mathrm{L}"'] \arrow[dr, dash, "\mathrm{M}"] & \\ P_1P_2\dots P_{k-1}L & & P_1P_2\dots P_{k-1}M \end{tikzcd}$$

if $(k-1)$-level is odd. To each admissible vertex $P^{(p-1)}C$ when we joint $C$ to the end of a block $P^{(p-1)}$ in $\mathcal{E}$ corresponds the symbol $A$ (or $B$) in the even or odd level in $\mathcal{T}$. Thus, to each admissible vertex $P^{(p-1)}C$ in $\mathcal{T}_{KS}$ corresponds an admissible vertex $(\tilde{P}^{(p-1)}A, \widetilde{\overline{P}}^{(p-1)}B)$ or $\tilde{P}^{(p-1)}B\bar{P}^{(p-1)}A$ in $\mathcal{D}_1$. Note that the admissibility in $\mathcal{E}$ corresponds to the admissibility in $\mathcal{T}$ since the $R$-parity in $\mathcal{E}$ corresponds to the $(R+L)$-parity in $\mathcal{T}$ and the shift $\sigma$ acting in $P^{(p-1)}C$ corresponds in $\mathcal{T}$ to a shift $\sigma$ acting in $\tilde{P}^{(p-1)}A$ or $\tilde{P}^{(p-1)}B\bar{P}^{(p-1)}A$. In this way, if $P^{(p-1)}C$ is admissible, that is,

$$\sigma^i(P^{(p-1)}C) \leq P^{(p-1)}C, \text{ for all } i$$

then, we also have that:

$$\sigma^i(\tilde{P}^{(p-1)}A) \leq \tilde{P}^{(p-1)}A, \text{ for all } i$$

or

$$\sigma^i(\tilde{P}^{(p-1)}B\bar{P}^{(p-1)}A) \leq \tilde{P}^{(p-1)}B\bar{P}^{(p-1)}A, \text{ for all } i\ .$$

∎

Consider now the Markov matrix associated to a sequence $S = \tilde{P}^{(p-1)}B\overline{P^{(p-1)}}A$ or $(\tilde{P}^{(p-1)}A, \overline{P^{(p-1)}}B)$ and denote by $d_P(t)$ the characteristic polynomial of the

Markov matrix $A_P$ where $P = P^{(p-1)}C \in \mathcal{T}_{KS}$ and corresponds to $\tilde{P} = \tilde{P}^{(p-1)}X$ in $\mathcal{D}_1$, where $X = A$ or $B$. Then the following result holds:

PROPOSITION 1. *To each $S = \tilde{P}^{(p-1)}B\overline{P^{(p-1)}}A$ or $(\tilde{P}^{(p-1)}A, \overline{P^{(p-1)}}B) \in \mathcal{D}_1$ there exist a decomposition of the matrix $A_S$ of the type:*

$$A_S = \begin{bmatrix} 1 & W_1 & W_2 \\ 0 & 0 & A_P \\ 0 & A_P & 0 \end{bmatrix}.$$

PROOF. Let $S = \tilde{P}^{(p-1)}B\tilde{\bar{P}}^{(p-1)}A$ or $(\tilde{P}^{(p-1)}A, \tilde{\bar{P}}^{(p-1)}B) \in \mathcal{D}_1$ then the Markov partition associated to $S$ is given by $\mathcal{P} = \mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$ where $\mathcal{P}_1 = \{I_i \colon 1 \leq i \leq p-1\}$, $\mathcal{P}_2 = I_p$, $\mathcal{P}_3 = \{I_i \colon p+1 \leq i \leq 2p-1\}$ and $\partial I_i = z_{i+1} - z_i$. When $S = (\tilde{P}^{(p-1)}B\ \tilde{\bar{P}}^{(p-1)}A)$ we have:

$$z_i \in J_1 = \{x_{2j} \colon 0 \leq j < p\} \text{ if } I_i \in \mathcal{P}_1$$

or

$$z_i \in J_2 = \{x_{2j+1} \colon 0 \leq j < p\} \text{ if } I_i \in \mathcal{P}_3$$

where $x_0$ (resp. $x_p$) corresponds to the critical point $c_1$ (resp. $c_2$). On the other hand, if $S = (\tilde{P}^{(p-1)}A, \tilde{\bar{P}}^{(p-1)}B)$ then:

$$z_i \in J_3 = \{x_{2j}, y_{2j} \colon 0 \leq j < \frac{p-2}{2}\} \text{ if } I_i \in \mathcal{P}_1$$

or

$$z_i \in J_4 = \{x_{2j+1}, y_{2j+1} \colon 0 \leq j < \frac{p-2}{2}\} \text{ if } I_i \in \mathcal{P}_3,$$

and in both cases $\mathcal{P}_2 = \{I_p\}$, with $\partial I_p = z_{p+1} - z_p$, where $z_p = \max\{J_1(\text{ or } J_3)\}$ and $z_{p+1} = \min\{J_2(\text{ or } J_4)\}$.Note also that if we look for the structure of $\mathcal{D}_1$ we conclude that: If $S = \tilde{P}^{(p-1)}B\bar{P}^{(p-1)}A \in \mathcal{D}_1$ then $S_{2i} \in \{L, A, M\}$, $S_{2i+1} \in \{M, B, R\}$ with $0 \leq i < p$. If $S = (\tilde{P}^{(p-1)}A, \bar{P}^{(p-1)}B) \in \mathcal{D}_1$ then $\tilde{P}_{2i} \in \{L, M\}$, $\tilde{P}_{2i+1} \in \{M, R\}$, $\bar{P}_{2i} \in \{M, R\}$ and $\bar{P}_{2i+1} \in \{L, M\}$ for $0 \leq i \leq \frac{p-2}{2}$. Thus, the even points establish a Markov shift and the odd points establish another Markov shift that is isomorphic to the previous one according to the symmetry of the cubic (where $S \in \mathcal{D}_1$). So, we only have to prove that each one of these shifts are isomorphic to the unimodal map associated $(A_{P^{(p)}}, \sigma_A)$. Note that, the even points are all smaller then the fixed point (that corresponds to the sequence of symbols $M^\infty$) whereas the odd points are all higher then the fixed point. So, we get two unimodal maps with critical points $c_1$ and $c_2$ given by sequences of symbols in $\{L, A, M\}$ or $\{M, B, R\}$ and by the admissibility unimodal rules. Thus, the partitions $\mathcal{P}_1$ and $\mathcal{P}_3$ are equivalent to the unimodal map associated and so they have the same Markov shifts. Finally, $\mathcal{P}_2$ introduce a state that transit to itself and also has transitions for other states that correspond to the transient part of the dynamics, $W$. ∎

COROLLARY 1. *To each $S = \tilde{P}^{(p-1)}B\overline{P^{(p-1)}}A$ or $(\tilde{P}^{(p-1)}A, \overline{P^{(p-1)}}B) \in \mathcal{D}_1$ there exist a decomposition of the characteristic polynomial, $d_S(t) = \det(I - t.A_S)$, associated to $A_S$ that is given by:*

$$d_S(t) = (1-t)d_P(t)d_P(-t)$$

*where $d_P(t) = \det(I - t.A_P)$.*

PROOF. Note that the decomposition of the characteristic polynomial follows from the previous decomposition of the Markov matrix. ∎

## 3. $\star$-Product operator

For the unimodal case it was defined the $\star$-product operator of symbolic sequences (see [**4**]). This product turns out to be a very useful tool to understand the properties of such maps.

In what follows, we extend the $\star$-product operator for the case of symbolic sequences associated to symmetric bimodal maps. Note that in $\mathcal{D}$ the $\star$-operation is consistent with the initial definition of the $\star$-product introduced by Derrida-Gervois-Pomeau for the unimodal case (see also [**10**], [**5**], [**12**] for the bimodal case).

According to Theorem 1, the tree $\mathcal{D}_1$ is isomorphic to $\mathcal{U}$ (ordered set of unimodal kneading sequences, see Fig. 3).

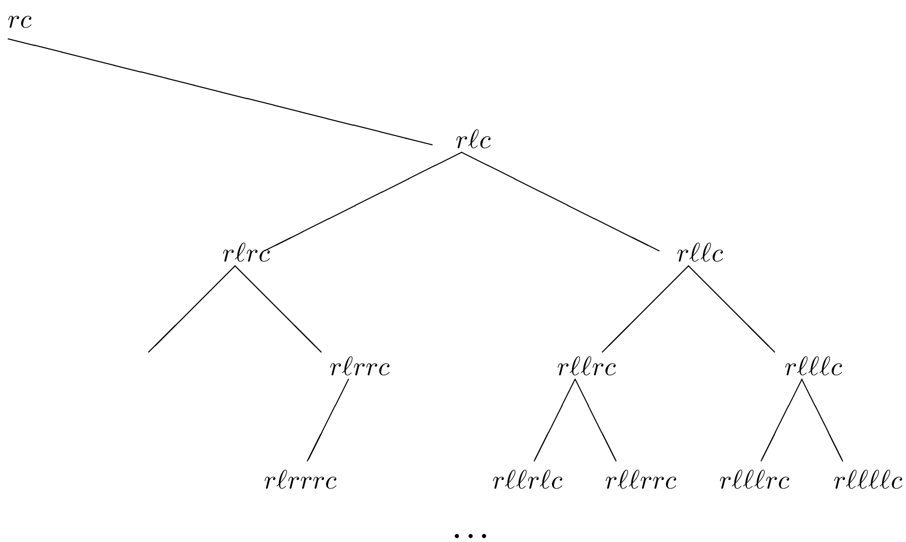



FIGURE 3. The tree $\mathcal{U}$ of unimodal kneading sequences.

From this set we can define another tree $\mathcal{F} = \{\mathcal{F}^- = \mathcal{U}\mathcal{U}$ , if the level is odd, or $\mathcal{F}^+ = (\mathcal{U}, \sigma(\mathcal{U}))$ if the level is even$\}$, see Fig. 4. Now using the symbolic codification applied to $f \circ f$ , with $f$ an unimodal map, we introduce the following translation rules:

$$\ell\ell \longrightarrow L, \ell c \longrightarrow A, \ell r \longrightarrow M, cr \longrightarrow B, rr \longrightarrow R, rc \longrightarrow C, r\ell \longrightarrow U.$$

By applying these rules, Fig. 4 can be rewritten as the tree $\mathcal{G} = \mathcal{G}^+ \cup \mathcal{G}^-$, see Fig. 5.

REMARK 1. *Let* $(x_1 \ldots x_{2n-1}c, \sigma(x_1 \ldots x_{2n-1}c)) \in \mathcal{F}^+$ *or* $x_1 \ldots x_{2n}cx_1 \ldots x_{2n}c \in \mathcal{F}^-$*, where* $x_i \in \{\ell, r\}$*, and* $\mathcal{F} = \mathcal{F}^+ \cup \mathcal{F}^-$ *is the tree presented in Figure 4. Let* $\mathcal{Q}$ *be the set of trimodal kneading data such that the image of both maxima are equal. With* $D = A$ *or* $B$*, we will write* $(P_{1\ldots}P_{2p-1}D, Q_1 \ldots Q_{2p-1}B, P_1 \ldots P_{2p-1}D) \in \mathcal{Q}$*, for even levels, and* $(P_{1\ldots}P_{2p-1}BQ_1 \ldots Q_{2p-2}A, P_1 \ldots P_{2p-1}B) \in \mathcal{Q}$*, for odd levels, with* $P_i, Q_j \in \{L, A, M, B, R, C, U\} = \{\ell\ell, \ell c, \ell r, cr, rr, rc, r\ell\}$*. With this notation, we can get* $\mathcal{G}$ *from the tree* $\mathcal{F}$*, (see also* [**9**]*).*

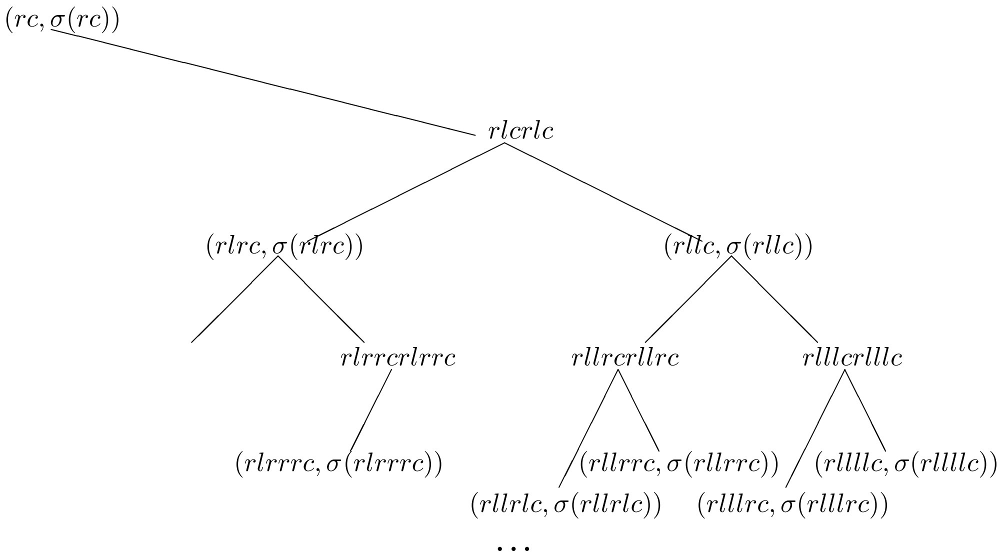



FIGURE 4. The tree $\mathcal{F}$

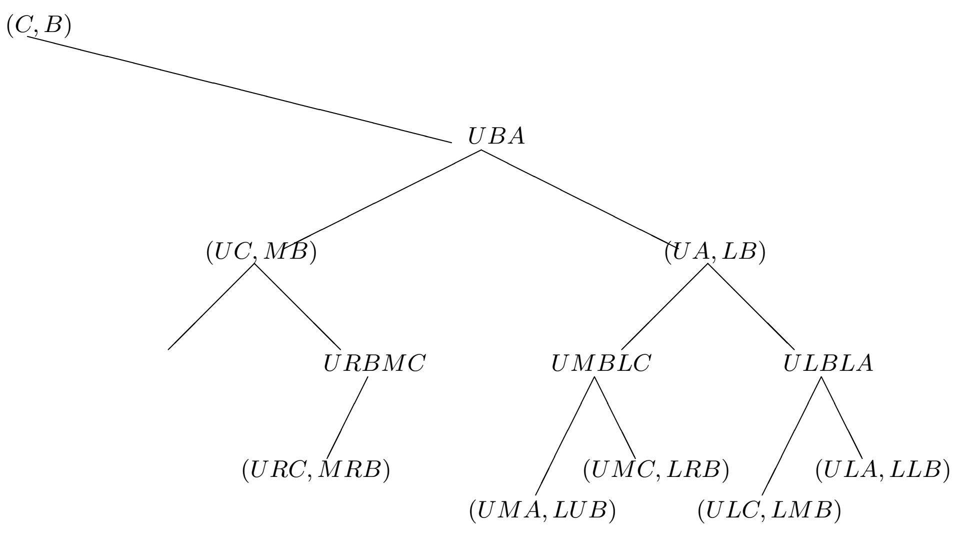



FIGURE 5. The tree $\mathcal{G}$ of the second factor for the star product.

Thus, we will consider the following different situations for the definition of the star product: first, let $F = (P, \overline{P}) \in \mathcal{D}$ and $G = (x_1 x_2 \dots x_{x-1} c, x_1 x_2 \dots x_{x-1} c)$, with $x_1 x_2 ... x_{x-1} c \in \mathcal{U}$; then, we let $F = PB\overline{P}A \in \mathcal{D}$ and $G \in \mathcal{G}$.

**Type 1.** Let $F = (P, \overline{P}) = (P^{(p-1)}A, \overline{P^{(p-1)}}B) \in \mathcal{D}$ be a bimodal kneading data and $G = (X, X)$ be a pair of unimodal kneading sequences. Then, we have

$$F \star G = (P, \overline{P}) \star (X, X) = (P^{(p-1)} \star X^{(x-1)} c, \overline{P^{(p-1)}} \star X^{(x-1)} c),$$

with
$$P^{(p-1)} \star X^{(x-1)}c = P^{(p-1)} A_1^{\pm} P^{(p-1)} A_2^{\pm} \dots P_{x-1}^{(p-1)} A_{x-1}^{\pm} P^{(p-1)} A,$$
where
$$A_i^{\pm} = \begin{cases} M \text{ if } x_i = r \\ A \text{ if } x_i = c \\ L \text{ if } x_i = \ell \end{cases} \text{if } P \text{ is even,} \quad A_i^{\pm} = \begin{cases} L \text{ if } x_i = r \\ A \text{ if } x_i = c \\ M \text{ if } x_i = \ell \end{cases} \text{if } P \text{ is odd.}$$
In a similar way,
$$\overline{P^{(p-1)}} \star X^{(x-1)}c = \overline{P^{(p-1)}} B_1^{\pm} \overline{P^{(p-1)}} B_2^{\pm} \dots \overline{P^{(p-1)}} B_{y-1}^{\pm} \overline{P^{(p-1)}} B,$$
where
$$B_i^{\pm} = \begin{cases} M \text{ if } x_i = r \\ B \text{ if } x_i = c \\ R \text{ if } x_i = \ell \end{cases} \text{if } \overline{P} \text{ is even,} \quad B_i^{\pm} = \begin{cases} R \text{ if } x_i = r \\ B \text{ if } x_i = c \\ M \text{ if } x_i = \ell \end{cases} \text{if } \overline{P} \text{ is odd.}$$
**Type 2.** Let $F = PB\overline{P}A = P^{(p-1)}B\overline{P^{(p-1)}}A \in \mathcal{D}$ and $G = X^{(x-1)}BY^{(y-1)}D \in \mathcal{G}^-$ (where $D = A$ or $C$) be two kneading data. Then,
$$F \star G = P^{(p-1)} B_1^{\pm} \overline{P^{(p-1)}} A_1^{\pm} P^{(p-1)} B_2^{\pm} \overline{P^{(p-1)}} A_2^{\pm} \dots P^{(p-1)} B \overline{P^{(p-1)}} A_x^{\pm} P^{(p-1)}$$
$$B_{x+1}^{\pm} \overline{P^{(p-1)}} A_{x+1}^{\pm} P^{(p-1)} B_{x+2}^{\pm} \dots \overline{P^{(p-1)}} A_{x+y-1}^{\pm} P^{(p-1)} B_{x+y}^{\pm} \overline{P^{(p-1)}} A.$$
Let $Z_i = X_i$, $Z_{x+i} = Y_i$ and $Z_{x+y} = D$ then
$$P^{(p-1)} B_i^{\pm} \overline{P^{(p-1)}} A_i^{\pm} = \begin{cases} P^{(p-1)} M \overline{P^{(p-1)}} M \text{ if } Z_i = L \\ P^{(p-1)} M \overline{P^{(p-1)}} A \text{ if } Z_i = A \\ P^{(p-1)} M \overline{P^{(p-1)}} L \text{ if } Z_i = M \\ P^{(p-1)} B \overline{P^{(p-1)}} L \text{ if } Z_i = B \\ P^{(p-1)} R \overline{P^{(p-1)}} L \text{ if } Z_i = R \\ P^{(p-1)} R \overline{P^{(p-1)}} A \text{ if } Z_i = C \\ P^{(p-1)} R \overline{P^{(p-1)}} M \text{ if } Z_i = U \end{cases} \text{if } P \text{ and } \overline{P} \text{ are even,}$$
$$P^{(p-1)} B_i^{\pm} \overline{P^{(p-1)}} A_i^{\pm} = \begin{cases} P^{(p-1)} R \overline{P^{(p-1)}} L \text{ if } Z_i = L \\ P^{(p-1)} R \overline{P^{(p-1)}} A \text{ if } Z_i = A \\ P^{(p-1)} R \overline{P^{(p-1)}} M \text{ if } Z_i = M \\ P^{(p-1)} B \overline{P^{(p-1)}} M \text{ if } Z_i = B \\ P^{(p-1)} M \overline{P^{(p-1)}} M \text{ if } Z_i = R \\ P^{(p-1)} M \overline{P^{(p-1)}} A \text{ if } Z_i = C \\ P^{(p-1)} M \overline{P^{(p-1)}} L \text{ if } Z_i = U \end{cases} \text{if } P \text{ and } \overline{P} \text{ are odd,}$$
**Type 3.** Let $F = P^{(p-1)}B\overline{P^{(p-1)}}A \in \mathcal{D}$ and $G = (X^{(n-1)}D, Y^{(n-1)}B) \in \mathcal{G}^+$ be two kneading data. Then
$$F \star G = (P^{(p-1)} B_1^{\pm} \overline{P^{(p-1)}} A_1^{\pm} P^{(p-1)} B_2^{\pm} \overline{P^{(p-1)}} A_2^{\pm} \dots P^{(p-1)} B_n^{\pm} \overline{P^{(p-1)}} A,$$
$$\overline{P^{(p-1)}} A_{n+1}^{\pm} P^{(p-1)} B_{n+1}^{\pm} \overline{P^{(p-1)}} A_{n+2}^{\pm} P^{(p-1)} B_{n+2}^{\pm} \dots \overline{P^{(p-1)}} A_{2n}^{\pm} P^{(p-1)} B).$$
The transformation rules are the same as above for the first sequence
$$F \star X^{(x-1)}A = P^{(p-1)} B_1^{\pm} \overline{P^{(p-1)}} A_1^{\pm} P^{(p-1)} B_2^{\pm} \overline{P^{(p-1)}} A_2^{\pm} \dots P^{(p-1)} B_n^{\pm} \overline{P^{(p-1)}} A,$$

except that $Z_i = B$ cannot occur. For the second position of the pair we have

$$\sigma^{p-1}(F) \star \sigma^{n-1}(Y^{(n-1)}B) = \\ B\overline{P^{(p-1)}}A^{\pm}_{n+1}P^{(p-1)}B^{\pm}_{n+1}\overline{P^{(p-1)}}A^{\pm}_{n+2}P^{(p-1)}B^{\pm}_{n+2}\dots\overline{P^{(p-1)}}A^{\pm}_{2n}P^{(p-1)}$$

$$\sigma(\sigma^{p-1}(F) \star \sigma^{n-1}(Y^{(n-1)}B)) = \\ \overline{P^{(p-1)}}A^{\pm}_{n+1}P^{(p-1)}B^{\pm}_{n+1}\overline{P^{(p-1)}}A^{\pm}_{n+2}P^{(p-1)}B^{\pm}_{n+2}\dots\overline{P^{(p-1)}}A^{\pm}_{2n}P^{(p-1)}B.$$

where

$$B^{\pm}_i\overline{P^{(p-1)}}A^{\pm}_iP^{(p-1)} = \begin{cases} M\overline{P^{(p-1)}}MP^{(p-1)} \text{ if } Y_i = L \\ M\overline{P^{(p-1)}}LP^{(p-1)} \text{ if } Y_i = M \\ B\overline{P^{(p-1)}}LP^{(p-1)} \text{ if } Y_i = B \\ R\overline{P^{(p-1)}}LP^{(p-1)} \text{ if } Y_i = R \\ R\overline{P^{(p-1)}}MP^{(p-1)} \text{ if } Y_i = U \end{cases} \quad \text{if } P \text{ and } \overline{P} \text{ are even,}$$

$$B^{\pm}_i\overline{P^{(p-1)}}A^{\pm}_iP^{(p-1)} = \begin{cases} R\overline{P^{(p-1)}}LP^{(p-1)} \text{ if } Y_i = L \\ R\overline{P^{(p-1)}}MP^{(p-1)} \text{ if } Y_i = M \\ B\overline{P^{(p-1)}}MP^{(p-1)} \text{ if } Y_i = B \\ M\overline{P^{(p-1)}}MP^{(p-1)} \text{ if } Y_i = R \\ M\overline{P^{(p-1)}}LP^{(p-1)} \text{ if } Y_i = U \end{cases} \quad \text{if } P \text{ and } \overline{P} \text{ are odd,}$$

The following examples illustrate the definitions given above:

Example 1.

$$(RMMA, LMMB) \star (r\ell c, r\ell c) = (RMMMRMMLRMMA, LMMMLMMRLMMB).$$

Example 2.

$$RBLA \star ULBLA = RRLMRMLMRBLLRMLMRMLA.$$

Example 3.

$$RBLA \star (UA, LB) = (RRLMRMLA, LLRMLMRB).$$

Remark 2. *Notice that, regarding Example 3, where both $P$ and $\overline{P}$ are even sequences, we have, for the second position of the pair,*

$$\sigma(\sigma(RBLA) \star \sigma(LB)) = \sigma(BLAR \star BL) = \sigma(BLLRMLMR) = LLRMLMRB.$$

Remark 3. *The star product in $\mathcal{D}$ is not a true binary product $A * B = C$, for all $A$ and $B \in \mathcal{D}$. When $C$ is factorizable, with $C \in \mathcal{D}$ then $B$ must be in $\mathcal{G}$.*

Remark 4. *Note that for all $A \in \mathcal{D}$ and $B \in \mathcal{G}$. then the result of the star product defined previous is also in $\mathcal{D}$.*

## 4. $\otimes$-Product between Markov matrices

In the same way, we can extend the $\otimes$-product between Markov matrices (introduced for unimodal maps in [**6**]) associated to symmetric bimodal maps, that is

$$A_S = A_V \otimes A_W$$

where $S = V \star W$ with $V \in \mathcal{D}$ and $W \in \mathcal{G}$.

THEOREM 2. *Let $V \in \mathcal{D}$ and $W \in \mathcal{G}$ then there exists a matrix product such that*

$$A_S = A_{V\star W} = A_V \otimes A_W$$

PROOF. It is based on a construction of a product on the matrices induced by the $\star$-product between kneading sequences. We will give this construction but only for the $\star$-product between kneading sequences of the first type. For the others it is technically similar and can be reproduced from this one. Let $W = (x_1x_2...x_{k-1}c, x_1x_2...x_{k-1}c) \in \mathcal{G}$. First of all note that the matrix $A_W$ is symmetric and so it can be written in the form:

$$A_W = \begin{bmatrix} 0 & 0 & \widehat{B}_X \\ N_1 & 1 & N_2 \\ B_X & 0 & 0 \end{bmatrix}$$

where $[N_1 \quad 1 \quad N_2]$ is the $k$ row and $[0 \quad 1 \quad 0]$ is the $k$ column. Denoting $B_X = [b_{ij}]$ then we define the $(k-1)\times(k-1)$ matrix $\hat{B}_X$, by $\hat{B}_X = B_X$ if $P^{(p)}$ is even and by $\hat{B}_X = [\hat{b}_{mj}]$ with $\hat{b}_{mj} = b_{ij}$ where $m = k - i$ if $P^{(p)}$ is odd. Given $V = (P, \overline{P}) \in \mathcal{D}$ and $W = (X, \overline{X}) \in \mathcal{G}$, it is immediate that its associated transition matrices, $A_V$ and $A_W$, are square, $(2p-1)$ and $(2k-1)$-dimensional matrices, respectively, where $p$ and $k$ denotes the number of symbols of the sequences $P$ and $X$. Analogously, it is fairly simple to see that the transition matrix associated with the sequence $V \star W$ is a square matrix with dimension $(2pk-1)$. Now, we need to show that the elements of $A_{V\star W}$ are completely determined by the knowledge of the matrices $A_V$ and $A_W$. Consider the symbolic shifts of the sequences $P\star X$ and $\overline{P}\star\overline{X}$ and denote the corresponding points of the interval by $p_i^j$ and $q_i^j$, that is, $p_i^j$ will be the point corresponding to the sequence $\sigma^{p(j-1)+(i-1)}(P\star X)$ and $q_i^j$ the point corresponding to the sequence $\sigma^{p(j-1)+(i-1)}(\overline{P}\star\overline{X})$. When one considers the collection of points of the interval from all the shifts cited above, we can see that they appear as groups of blocks of $x$ points. Considering the order of the shifted sequences $\sigma^i(P)$, $\sigma^j(\overline{P})$, $\sigma^k(X)$ and $\sigma^n(\overline{X})$ and the way those sequences appear as subsets of the partition induced by the sequence $V \star W$, we can conclude (see also [**6**]) that the matrix $A_{V\star W}$ has the following block structure:

$$\begin{bmatrix}
A_{1,1} & A_{1,2} & \cdots & A_{1,l+m+r} & N_1 \\
\vdots & \vdots & & \vdots & \vdots \\
A_{l,1} & A_{l,2} & \cdots & A_{l,l+m+r} & N_l \\
Z_{l+1,1} & Z_{l+1,2} & \cdots & Z_{l+1,l+m+r} & \bar{A}_X \\
A_{l+2,1} & A_{l+2,2} & \cdots & A_{l+2,l+m+r} & N_{l+2} \\
\vdots & \vdots & & \vdots & \vdots \\
A_{l+m+1,1} & A_{l+m+1,2} & \cdots & A_{l+m+1,l+m+r} & N_{l+m+1} \\
\widetilde{A}_X & Z_{l+m+2,2} & \cdots & Z_{l+m+2,l+m+r} & N_{l+m+2} \\
N_{l+m+3} & A_{l+m+3,2} & \cdots & A_{l+m+3,l+m+r} & A_{l+m+3,l+m+r+1} \\
\vdots & \vdots & & \vdots & \vdots \\
N_{l+m+r+2} & A_{l+m+r+2,2} & \cdots & A_{l+m+r+2,l+m+r} & A_{l+m+r+2,l+m+r+1}
\end{bmatrix},$$

where $l, m$ and $r$ are, respectively, the number of symbols $L, M$ and $R$ in the sequence $V$, and $A_{i,j}$, with $(i,j) \neq (l+m+1, 1)$, is either one of these $k \times k$

matrices

$$\begin{bmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix},$$
$$\begin{bmatrix} 0 & \cdots & 0 & 1 \\ 0 & \cdots & 1 & 0 \\ \vdots & & \vdots & \vdots \\ 1 & \cdots & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 1 & 1 & \cdots & 1 \end{bmatrix},$$

or a null block, and

$$A_{l+m+1,1} = \begin{bmatrix} n_1 & \cdots & n_y & n_{y+1} \\ b_{1,1} & \cdots & b_{1,y} & m_1 \\ \vdots & & \vdots & \vdots \\ b_{y,1} & \cdots & b_{y,y} & m_y \end{bmatrix},$$

where the submatrix $[b_{i,j}]$ is defined by the matrix $B_X$. The matrices $Z_k$ and $N_j$ are null matrices, except $N_l$, $N_{l+1}$ and $N_{l+m+2}$ that can contain some elements 1. The distribution of the previous blocks $A_{i,j}$, with $(i,j) \neq (l+m+1,1)$, is given by the structure of the matrix $A_V$. On the other hand, the internal structure of each block $A_{i,j}$ is determined by the order of the shifts of the sequence $W$. For the case of the block $A_{l+m+1,1}$, its submatrix $[b_{i,j}]$ has an internal structure determined by $B_X$. The elements $n_i$ and $m_j$ are null except those needed to preserve the continuity of the transitions (from the fact that $f$ is a continuous function). Analogously, the block $\bar{A}_X$ is determined by $\widehat{B}_X$. Finally, the blocks $N_l$, $N_{l+1}$, $N_{l+m+2}$ are null except, once again, for the elements needed to preserve the continuity of the transitions. ∎

The following example illustrate the use of the previous theorem.

Example 4. *Let $V = (RMMA, LMMB)$ and $W = (r\ell c, r\ell c)$. Then, we have $S = V \star W = (RMMMRMMLRMMA, LMMMLMMRLMMB)$, and the $\otimes$-product of its matrices,*

$$A_V = \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}; \qquad A_W = \left( \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} \right),$$

$A_S = A_{V\star W}$, *is given by*

$$\begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}$$

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